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HOMOGENEOUS PROBLEM FOR A WEDGE WITH A SYMMETRIC CRACK AT THE APEX*

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An exact solution of the plane, static homogeneous problem of the theory of elasticity of class N[1], is constructed for a wedge, encompassing a half-space, with a crack at the apex situated in the bisector plane.

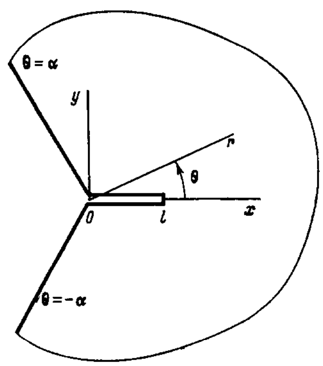

Fig.1

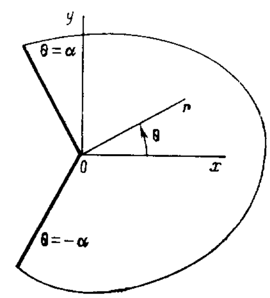

Fig.2

1. Formulation of the problem. We consider an infinite elastic wedge with the cone angle greater than π, containing a crack at $y=0, x<l$ (Fig.1). The wedge and crack edges are stress-free. We assume that at infinity the solution of the corresponding symmetrical (with respect to the plane $\theta=0$) problem behaves in the same manner, as the solution of the symmetric problem for a wedge $-\alpha\leqslant\theta\leqslant\alpha, \pi/2<\alpha<\pi$ with stress-free edges (Fig.2), asymptotically largest at infinity and satisfying the condition that the stresses decay at infinity. As we know (see e.g. /1/), the latter solution has the form ($\sigma_\theta, \tau_{r\theta}, \sigma_r$ are the stresses and u_θ, u_r the displacements)

$$\begin{aligned}
&\sigma_\theta = A_{\rm I}\,[(\lambda_1+1)\cos(\lambda_1-1)\,\theta - f_1(\alpha)\cos(\lambda_1+1)\,\theta]\\
&\tau_{r\theta} = A_{\rm I}\,[(\lambda_1-1)\sin(\lambda_1-1)\,\theta - f_1(\alpha)\sin(\lambda_1+1)\,\theta]\\
&\sigma_r = A_{\rm I}\,[(3-\lambda_1)\cos(\lambda_1-1)\,\theta + f_1(\alpha)\cos(\lambda_1+1)\,\theta]\\
&f_1(\alpha)=\frac{(\lambda_1-1)\sin(\lambda_1-1)\,\alpha}{\sin(\lambda_1+1)\,\alpha},\quad A_{\rm I}=\frac{C_{\rm I}}{2}(2\pi r)^{\lambda_1-1}\\
&-\alpha\leqslant\theta\leqslant\alpha,\ \pi/2<\alpha<\pi
\end{aligned}\tag{1.1}$$

Here $\lambda_1(\alpha)\in(1/2, 1)$ is a unique root of the equation $\sin 2p\alpha+p\sin 2\alpha=0$ in the strip $0<\operatorname{Re}p<1$ of the complex plane p, and $C_{\rm I}$ is an arbitrary real constant.

The solution (1.1) must be realized in the form of an asymptotics at infinity of the required solution of the initial symmetric problem for a wedge with a crack. Thus the boundary conditions of the initial symmetric problem can be written in the form

$$\theta=\alpha,\ \sigma_\theta=\tau_{r\theta}=0\ (\pi/2<\alpha<\pi)\tag{1.2}$$

$$\begin{aligned}
&\theta=0,\ \tau_{r\theta}=0\\
&\theta=0,\ r<l,\ \sigma_\theta=0;\ \theta=0,\ r>l,\ u_\theta=0
\end{aligned}\tag{1.3}$$

Moreover, the following condition, in particular, must hold at infinity:

$$\theta=0,\quad r\to\infty,\quad \sigma_\theta\sim\frac{Q_{\rm I}}{r^{1-\lambda_1}}$$

$$Q_{\rm I}=C_{\rm I}(2\pi)^{\lambda_1-1}\frac{\lambda_1\cos\lambda_1\alpha\sin\alpha+\sin\lambda_1\alpha\cos\alpha}{\sin(\lambda_1+1)\,\alpha}$$

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where the constant C_I is assumed given by the condition of the problem. The constant has a dimension of force divided by the length to power $\lambda_1(\alpha)+1$.

We have the following asymptotics /1/ near the crack end:

$$\theta=0,\quad r\to l+0,\quad \sigma_\theta\sim\frac{K_I}{\sqrt{2\pi(r-l)}}$$
$$\theta=+0,\quad r\to l-0,\quad \frac{\partial u_\theta}{\partial r}\sim-\frac{2(1-\nu^2)}{E}\frac{K_I}{\sqrt{2\pi(r-l)}}$$

where K_I is the stress intensity coefficient, to be determined, E is the Young's modulus and ν is the Poisson's ratio. If the wedge angle does not exceed π, then the symmetric homogeneous problem in question belongs to class S[1] and has a trivial solution only.

In dealing with a skew symmetric homogeneous problem we shall assume that the wedge angle varies within the limits $2\alpha_*<2\alpha<2\pi$ $(2\alpha_*\approx 257°;\ \alpha_*$ is a unique root of the equation $2\alpha\cos 2\alpha-\sin 2\alpha=0$ on the interval $\pi/2<\alpha<\pi)$, since in the case of $2\alpha\leqslant 2\alpha_*$ the problem belongs to class S and has a trivial solution only.

The solution of the skew symmetric problem for the wedge $-\alpha\leqslant\theta\leqslant\alpha,\ \alpha_*<\alpha<\pi$ with stress-free edges, asymptotically largest at infinity and satisfying the condition that the stress decays at infinity, has the form

$$\sigma_\theta=A_{II}[(\lambda_2+1)\sin(\lambda_2-1)\theta-f_2(\alpha)\sin(\lambda_2+1)\theta] \tag{1.4}$$
$$\tau_{r\theta}=A_{II}[-(\lambda_2-1)\cos(\lambda_2-1)\theta+f_2(\alpha)\cos(\lambda_2+1)\theta]$$
$$\sigma_r=A_{II}[(3-\lambda_2)\sin(\lambda_2-1)\theta+f_2(\alpha)\sin(\lambda_2+1)\theta]$$
$$f_2(\alpha)=\frac{(\lambda_2+1)\sin(\lambda_2-1)\alpha}{\sin(\lambda_2+1)\alpha},\quad A_{II}=\frac{C_{II}}{2}(2\pi r)^{\lambda_2-1}$$
$$-\alpha\leqslant\theta\leqslant\alpha,\ \alpha_*<\alpha<\pi$$

Here $\lambda_2(\alpha)\in(1/2,1)$ is a unique root of the equation $\sin 2p\alpha-p\sin 2\alpha=0$ in the strip $0<\operatorname{Re}p<1$, and C_{II} is an arbitrary real constant. The solution (1.4) must be realized in the form of an asymptotics at infinity of the solution of the skew symmetric problem in question.

We write the boundary conditions of the skew symmetric problem in the form

$$\theta=\alpha,\ \sigma_\theta=\tau_{r\theta}=0\ (\alpha_*<\alpha<\pi),\qquad \theta=0,\ \sigma_\theta=0$$
$$\theta=0,r<l,\ \tau_{r\theta}=0;\ \theta=0,r>l,\ u_r=0,\quad \theta=0,\quad r\to\infty,\quad \tau_{r\theta}\sim\frac{Q_{II}}{r^{1-\lambda_2}}$$
$$Q_{II}=C_{II}(2\pi)^{\lambda_2-1}\frac{-\lambda_2\cos\lambda_2\alpha\sin\alpha+\sin\lambda_2\alpha\cos\alpha}{\sin(\lambda_2+1)\alpha}$$

The constant C_{II} is assumed given. It has the dimension of force divided by length to power $\lambda_2(\alpha)+1$. Let us write the asymptotics near the crack end

$$\theta=0,\quad r\to l+0,\quad \tau_{r\theta}\sim\frac{K_{II}}{\sqrt{2\pi(r-l)}}$$
$$\theta=+0,\quad r\to l-0,\quad \frac{\partial u_r}{\partial r}\sim\frac{2(1-\nu^2)}{E}\frac{K_{II}}{\sqrt{2\pi(l-r)}}$$

where K_{II} is the stress intensity coefficient, to be determined. We can assume without loss of generality that the crack is of unit length.

2. Wiener—Hopf equations and stress intensity coefficients. The solution of the symmetric homogeneous problem in question is a sum of solutions of the following two problems.

In the first problem the condition (1.2) and second condition of (1.3) are retained, and the first condition of (1.3) is replaced by

$$\theta=0,\quad r<1,\quad \sigma_\theta=-\frac{Q_I}{r^{1-\lambda_1}} \tag{2.1}$$

The stress decay at infinity as $o(1/r)$. The second problem is a symmetric problem for a wedge with stress-free edges (its solution (1.1) will be kept in mind).

Applying the Mellin integral transform to the equations of equilibrium, condition of compatibility of deformations, Hooke's law and conditions (1.2) and taking into account the second condition of (1.3) and condition (2.1), we arrive at the following functional Wiener—Hopf equation:

$$\Phi_1^+(p)-\frac{Q_I}{p+\lambda_1}=-\operatorname{tg}p\pi G_1(p)\Phi_1^-(p) \tag{2.2}$$
$$G_1(p)=\frac{2(p^2\sin^2\alpha-\sin^2 p\alpha)}{-\operatorname{tg}p\pi(\sin 2p\alpha+p\sin 2\alpha)}$$

$$\Phi_1^+(p) = \int_1^\infty \sigma_\theta(r, 0)\, r^p\, dr, \quad \Phi_1^-(p) = \frac{E}{2(1-\nu^2)} \int_0^1 \frac{\partial u_\theta}{\partial r}\bigg|_{\theta=+0} r^p\, dr$$

Using the factorization /2,3/

$$G_1(p) = \frac{G_1^+(p)}{G_1^-(p)} \quad (\mathrm{Re}\, p = 0) \tag{2.3}$$

$$\exp\left[\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\ln G_1(t)}{t-p}\, dt\right] = \begin{cases} G_1^+(p), & \mathrm{Re}\, p < 0 \\ G_1^-(p), & \mathrm{Re}\, p > 0 \end{cases}$$

$$p \operatorname{ctg} p\pi = K^+(p)\, K^-(p), \quad K^\pm(p) = \frac{\Gamma(1 \mp p)}{\Gamma(1/2 \mp p)} \tag{2.4}$$

where ($\Gamma(z)$ is the Euler gamma function, we can write the equation (2.2) in the form

$$\frac{K^+(p)\,\Phi_1^+(p)}{pG_1^+(p)} - \frac{Q_\mathrm{I} K^+(p)}{p(p+\lambda_1)\, G_1^+(p)} = -\frac{\Phi_1^-(p)}{K^-(p)\, G_1^-(p)} \quad (\mathrm{Re}\, p = 0) \tag{2.5}$$

Using the expression

$$\frac{Q_\mathrm{I} K^+(p)}{p(p+\lambda_1)\, G_1^+(p)} = \frac{Q_\mathrm{I}}{p+\lambda_1}\left[\frac{K^+(p)}{pG_1^+(p)} + \frac{K^+(-\lambda_1)}{\lambda_1 G_1^+(-\lambda_1)}\right] - \frac{Q_\mathrm{I} K^+(-\lambda_1)}{\lambda_1(p+\lambda_1)\, G_1^+(-\lambda_1)} \quad (\mathrm{Re}\, p = 0)$$

we obtain, in accordance with (2.5),

$$\frac{K^+(p)\,\Phi_1^+(p)}{pG_1^+(p)} - \frac{Q_\mathrm{I}}{p+\lambda_1}\left[\frac{K^+(p)}{pG_1^+(p)} + \frac{K^+(-\lambda_1)}{\lambda_1 G_1^+(-\lambda_1)}\right] = -\frac{\Phi_1^-(p)}{K^-(p)\, G_1^-(p)} - \frac{Q_\mathrm{I} K^+(-\lambda_1)}{\lambda_1(p+\lambda_1)\, G_1^+(-\lambda_1)} \quad (\mathrm{Re}\, p = 0) \tag{2.6}$$

The function appearing in the left-hand side of (2.6) is analytic in the half-plane $\mathrm{Re}\, p < 0$, while the function in its right-hand side is analytic in the half-plane $\mathrm{Re}\, p > 0$. It follows therefore that both functions are equal to a single function analytic in the whole p-plane. We have the following asymptotic relations ($p \to \sim$):

$$\Phi_1^+(p) \sim \frac{K_\mathrm{I}}{\sqrt{-2p}}, \quad \Phi_1^-(p) \sim -\frac{K_\mathrm{I}}{\sqrt{2p}} \tag{2.7}$$

From (2.3), (2.4) and (2.7) it follows that the functions in the left-and right-hand sides of (2.6) tend to zero as $p \to \infty$ in the half-plane $\mathrm{Re}\, p<0$ and $\mathrm{Re}\, p>0$ respectively. Thus the single analytic function is identically equal to zero in the whole p-plane. A solution of the functional equation (2.2) has the form

$$\Phi_1^+(p) = \frac{pQ_\mathrm{I}}{p+\lambda_1}\left[\frac{K^+(p)}{pG_1^+(p)} + \frac{K^+(-\lambda_1)}{\lambda_1 G_1^+(-\lambda_1)}\right]\frac{G_1^+(p)}{K^+(p)} \quad (\mathrm{Re}\, p < 0)$$

$$\Phi_1^-(p) = -\frac{Q_\mathrm{I} K^+(-\lambda_1)}{\lambda_1(p+\lambda_1)\, G_1^+(-\lambda_1)}\, G_1^-(p)\, K^-(p) \quad (\mathrm{Re}\, p > 0)$$

We construct the solution of the skew symmetric problem is exactly the same manner, and the following functional equation corresponds to this solution:

$$\Phi_2^+(p) - \frac{Q_\mathrm{II}}{p+\lambda_2} = -\operatorname{tg} p\pi G_2(p)\, \Phi_2^-(p)$$

$$G_2(p) = \frac{2(p^2 \sin^2\alpha - \sin^2 p\alpha)}{-\operatorname{tg} p\pi\, (\sin 2p\alpha - p \sin 2\alpha)}$$

$$\Phi_2^+(p) = \int_1^\infty \tau_{r\theta}(r, 0)\, r^p\, dr, \quad \Phi_2^-(p) = \frac{E}{2(1-\nu^2)}\int_0^1 \frac{\partial u_r}{\partial r}\bigg|_{\theta=+0} r^p\, dr$$

The factorization (2.3) in which $G_1(p), G_1^\pm(p)$ is replaced by $G_2(p), G_2^\pm(p)$, holds for the function $G_2(p)$. The stress intensity coefficients in the initial homogeneous solution are expressed by the formula

$$K_n = \frac{\sqrt{2}\, K^+(-\lambda_k)}{\lambda_k G_k^+(-\lambda_k)}\, Q_n l^{\lambda_k - 1/2}$$

where $k = 1,2$; $n = \mathrm{I}$ when $k = 1$ and $n = \mathrm{II}$ when $k = 2$.

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